Combining Diffusion And Jump Size Variances

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In a jump diffusion equation for stock price both the expected return and jump size are normally-distributed random variables with a mean and a variance. In this white paper we will combine the variances associated with expected return and jump size into one normally-distributed random variable. We do this so that we can use the Black-Scholes option pricing model when pricing options on jump diffusion processes. To that end we will work through the following hypothetical problem...

Our Hypothetical Problem

We are tasked with building a model to forecast ABC Company stock price given the following go-forward model assumptions...

Table 1: Go-Forward Model Assumptions

Symbol	Description	Value
S_0	Stock price at time zero ($)	20.00
μ	Expected return mean (%)	12.50
σ	Expected return volatility (%)	35.00
ω	Jump size mean (%)	3.50
υ	Jump size volatility (%)	10.00
λ	Average number of annual jumps (#)	3.00
t	Time in years (#)	4.00

Our task is to answer the following questions...

Question 1: What is random stock price at the end of year 4 given that there were k = 10 jumps drawn from a Poisson distribution and z = -1.20 drawn from a normal distribution.

Question 2: What is expected conditional stock price at the end of year 4 given that there were 10 jumps over the time period $[0, 4]$?

Question 3: What is expected unconditional stock price at the end of year 4?

An Equation For Combined Variance

We defined the independent normally-distributed random variables x and y with the following means and variances...

$$x \sim N\bigg[0, 1\bigg] \text{ ...and... } y \sim N\bigg[0, 1\bigg] \tag{1}$$

Because the random variables x and y in Equation (1) above are standardized, independent and normally-distributed, we can make the following statements as to expectations...

$$\mathbb{E}\bigg[x\bigg] = 0 \text{ ...and... } \mathbb{E}\bigg[x^2\bigg] = 1 \text{ ...and... } \mathbb{E}\bigg[y\bigg] = 0 \text{ ...and... } \mathbb{E}\bigg[y^2\bigg] = 1 \text{ ...and... } \mathbb{E}\bigg[x\,y\bigg] = 0 \tag{2}$$

Using Equations (1) and (2) above we defined the jump diffusion equation for conditional random stock price as follows... [2]

$$S(k)_t = S_0 \operatorname{Exp}\left\{\mu\,t - \lambda\,\omega\,t + k\,\ln(1+\omega) - \frac{1}{2}\,\sigma^2\,t - k\,\frac{1}{2}\,\upsilon^2 + \sigma\,\sqrt{t}\,x + \upsilon\,\sqrt{k}\,y\right\} \tag{3}$$

Because the random variables x and y in Equation (3) above are normally-distributed then the following function $f(x,y)$ is also normally-distributed...

$$f(x,y)=\sigma\,\sqrt{t}\,x+\upsilon\,\sqrt{k}\,y\ \text{ ...and... }\ f(x,y)\sim N\left[\text{mean},\text{variance}\right] \tag{4}$$

Using Equation (2) above the equation for the first moment of Equation (4) above is...

$$\begin{aligned}\mathbb{E}\left[f(x,y)\right]&=\mathbb{E}\left[\sigma\,\sqrt{t}\,x+\upsilon\,\sqrt{k}\,y\right]\\&=\sigma\,\sqrt{t}\,\mathbb{E}\left[x\right]+\upsilon\,\sqrt{k}\,\mathbb{E}\left[y\right]\\&=0\ \text{ ...because... }\ \mathbb{E}\left[x\right]=0\ \text{ ...and... }\ \mathbb{E}\left[y\right]=0\end{aligned} \tag{5}$$

Using Equation (2) above the equation for the second moment of Equation (4) above is..

$$\begin{aligned}\mathbb{E}\left[f(x,y)^2\right]&=\mathbb{E}\left[\sigma^2 t\,x^2+\upsilon^2 k\,y^2+2\,\sigma\sqrt{t}\,\upsilon\sqrt{k}\,x\,y\right]\\&=\sigma^2 t\,\mathbb{E}\left[x^2\right]+\upsilon^2 k\,\mathbb{E}\left[y^2\right]+2\,\sigma\sqrt{t}\,\upsilon\sqrt{k}\,\mathbb{E}\left[x\,y\right]\\&=\sigma^2 t+\upsilon^2 k\ \text{ ...because... }\ \mathbb{E}\left[x^2\right]=1\ \text{ ...and... }\ \mathbb{E}\left[y^2\right]=1\ \text{ ...and... }\ \mathbb{E}\left[x\,y\right]=0\end{aligned} \tag{6}$$

Using Equations (5) and (6) above the mean and variance of Equation (4) above are...

$$\text{mean}=\mathbb{E}\left[f(x,y)\right]=0\ \text{ ...and... }\ \text{variance}=\mathbb{E}\left[f(x,y)^2\right]-\left[\mathbb{E}\left[f(x,y)\right]\right]^2=\sigma^2 t+\upsilon^2 k \tag{7}$$

We will define the random variable z to be normally-distributed with mean zero and variance one. Using Equation (7) above we can rewrite Equation (4) above as...

$$f(x,y)=\text{mean}+\sqrt{\text{variance}}\,z=0+\sqrt{\sigma^2 t+\upsilon^2 k}\,z=\sqrt{\sigma^2 t+\upsilon^2 k}\,z\ \text{ ...where... }\ z\sim N\left[0,1\right] \tag{8}$$

Stock Price Equations

Using Equation (8) above we can rewrite conditional random stock price Equation (3) above as...

$$S(k)_t=S_0\,\mathrm{Exp}\left\{\mu\,t-\lambda\,\omega\,t+k\,\ln(1+\omega)-\frac{1}{2}\,\sigma^2\,t-k\,\frac{1}{2}\,\upsilon^2+\sqrt{\sigma^2 t+\upsilon^2 k}\,z\right\} \tag{9}$$

We will make the following variable definition...

$$\text{if... }\ \hat{\sigma}=\sqrt{\sigma^2+\frac{\upsilon^2 k}{t}}\ \text{ ...then... }\ -\frac{1}{2}\,\hat{\sigma}^2 t=-\frac{1}{2}\,\sigma^2\,t-k\,\frac{1}{2}\,\upsilon^2\ \text{ ...and... }\ \hat{\sigma}\,\sqrt{t}=\sqrt{\hat{\sigma}^2 t}=\sqrt{\sigma^2 t+\upsilon^2 k} \tag{10}$$

Using the definitions in Equation (10) above we can rewrite conditional random stock price Equation (9) above as...

$$\begin{aligned}S(k)_t&=S_0\,\mathrm{Exp}\left\{\mu\,t-\lambda\,\omega\,t+k\,\ln(1+\omega)-\frac{1}{2}\,\hat{\sigma}^2\,t+\hat{\sigma}\,\sqrt{t}\,z\right\}\\&=S_0\mathrm{Exp}\left\{k\,\ln(1+\omega)\right\}\left(\mu-\lambda\,\omega-\frac{1}{2}\,\hat{\sigma}^2\right)t+\hat{\sigma}\,\sqrt{t}\,z\Bigg\}\\&=S_0\left(1+\omega\right)^k\mathrm{Exp}\left\{\left(\mu-\lambda\,\omega-\frac{1}{2}\,\hat{\sigma}^2\right)t+\hat{\sigma}\,\sqrt{t}\,z\right\}\end{aligned} \tag{11}$$

We will make the following variable definition...

$$\text{if... }\ \theta=\left(\mu-\lambda\,\omega-\frac{1}{2}\,\hat{\sigma}^2\right)t+\hat{\sigma}\,\sqrt{t}\,z\ \text{ ...then... }\ \theta\sim N\left[\left(\mu-\lambda\,\omega-\frac{1}{2}\,\hat{\sigma}^2\right)t,\hat{\sigma}^2 t\right] \tag{12}$$

Using Equation (12) above we can rewrite conditional random stock price Equation (11) above as...

$$S(k)_t = S_0 \left(1+\omega\right)^k \mathrm{Exp}\left\{\theta\right\} \tag{13}$$

Given that the variable θ in Equation (13) above is normally-distributed then the exponential of θ is lognormally-distributed. Using Equations (12) and (13) above the equation for expected conditional stock price at time t is...

$$\begin{aligned}\mathbb{E}\left[S(k)_t\right] &= \mathbb{E}\left[S_0 \left(1+\omega\right)^k \mathrm{Exp}\left\{\theta\right\}\right]\\ &= S_0 \left(1+\omega\right)^k \mathrm{Exp}\left\{\text{mean} + \frac{1}{2}\,\text{variance}\right\}\\ &= S_0 \left(1+\omega\right)^k \mathrm{Exp}\left\{\left(\mu - \lambda\,\omega - \frac{1}{2}\,\hat{\sigma}^2\right)t + \frac{1}{2}\,\hat{\sigma}^2 t\right\}\\ &= S_0 \left(1+\omega\right)^k \mathrm{Exp}\left\{\mu\,t - \lambda\,\omega\,t\right\}\end{aligned} \tag{14}$$

Using Equation (14) above the equation for expected unconditional stock price at time t is... [1]

$$\mathbb{E}\left[S_t\right] = \sum_{k=0}^{\infty} \frac{(\lambda\,t)^k}{k!}\,\mathrm{Exp}\left\{-\lambda\,t\right\}\mathbb{E}\left[S(k)_t\right] = S_0\,\mathrm{Exp}\left\{\mu\,t\right\} \tag{15}$$

The Answers To Our Hypothetical Problem

Question 1: What is random stock price at the end of year 4 given that there were k = 10 jumps drawn from a Poisson distribution and z = -1.20 drawn from a normal distribution.

Using Equation (10) above and the data in Table 1 above the equation for combined volatility is...

$$\hat{\sigma} = \sqrt{\sigma^2 + \frac{\upsilon^2 k}{t}} = \sqrt{0.35^2 + \frac{0.10^2 \times 10}{4}} = 0.38406 \tag{16}$$

Using Equations (11) and (16) above and the data in Table 1 above the answer to the question is...

$$S(10)_4 = 20.00 \times \left(1+0.035\right)^{10} \times \mathrm{Exp}\left\{\left(0.125 - 3\times 0.035 - \frac{1}{2}\times 0.38406^2\right)\times 4 + 0.38406 \times \sqrt{4} \times -1.2\right\} = 9.05 \tag{17}$$

Question 2: What is expected conditional stock price at the end of year 4 given that there were 10 jumps over the time period $[0, 4]$?

Using Equation (14) above and the data in Table 1 above the answer to the question is...

$$\mathbb{E}\left[S(10)_4\right] = 20.00 \times \left(1+0.035\right)^{10} \times \mathrm{Exp}\left\{0.125 \times 4 - 3 \times 0.035 \times 4\right\} = 30.56 \tag{18}$$

Question 3: What is expected unconditional stock price at the end of year 4?

Using Equation (15) above and the data in Table 1 above the answer to the question is...

$$\mathbb{E}\left[S_4\right] = 20.00 \times \mathrm{Exp}\left\{0.125 \times 4\right\} = 32.97 \tag{19}$$

References

[1] Gary Schurman, *The Compensated Poisson Process,* March, 2021.

[2] Gary Schurman, *A Jump Diffusion Model For Stock Price,* March, 2021.